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CONTROL SYNTHESIS IN THE PROBLEM OF THE TIME-OPTIMAL INTERSECTION OF A SPHERE†

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The classical problem of the time-optimal control of the motion of a point mass is considered, with the control accomplished by applying a force of limited magnitude. Open- and closed-loop control laws are constructed that ensure intersection with a sphere (from outside or inside) in a coordinate space of arbitrary dimensions. The maximum principle is used to show that the control is of maximum magnitude and maintains a constant direction, while the optimal trajectories are parabolas; the general situation of a multi-dimensional space is equivalent to the two-dimensional (plane) case. The feedback controls and optimal time of motion are constructed as functions of the phase coordinates, the Bellman function of the problem is analysed in detail, qualitative properties of the control and the optimal time of motion in the neighbourhood of the critical phase states of the system are determined. Mathematical simulation is used to construct a section of the Bellman function over a large range of motion parameters; the results are compared with those of analytic and asymptotic studies.

Time-optimal control of the motion of a point mass by bounded and impulse forces, in a variety of formulations, is one of the basic models in applications of optimal control theory and methods [1–5], as well as in other areas.

The problem of time-optimal intersection with a sphere of arbitrary radius, defined in a geometrical space (disregarding velocity), is of interest in both theoretical and applied aspects. It provides a good approximate model for investigating such space flight problems as a controlled object entering the "sphere of attraction" of a celestial body or escaping from that sphere. It may also be considered as an extension of the problem of reaching a fixed geometrical point to the problem of reaching the neighbourhood of the origin or of a point target moving at a fixed velocity.

1. FORMULATION OF THE PROBLEM

We will consider an optimal control problem of the following form [1, 5]

$$\begin{gathered} \dot{x} = v, \quad \dot{v} = u, \quad x, v, u \in E^N, \quad x(0) = x^0, \quad v(0) = v^0 \\ x(t_f) \in S_r, \quad S_r = \{x : |x| = r\}, \quad t_f \to \min, \quad |u| \leq u_0 \end{gathered} \tag{1.1}$$

We assume, without loss of generality, that the sphere is centred at the origin. Problem (1.1) will be investigated in a situation of general position, when $0 < u_0, r < \infty$. Introduction of dimensionless variables by using the length r and the time $\tau = (r/\mathbf{u}_0)^{1/2}$ as scales yields Eqs (1.1) with $r = 1$, $u_0 = 1$. Thus simplified, the system involves no parameters, and the solution will be defined by arbitrary values of the n-vectors x^0, v^0. Note that the point x^0 may be situated either inside ($|x^0| < 1$) or outside ($|x^0| > 1$) the spherical region B bounded by the unit sphere S_1 ($|x| = 1$).

A solution of the control problem exists for arbitrary values of x^0, $v^0 \in E^n$. To construct optimal control laws we will apply the necessary conditions of the maximum principle [1]. We introduce variables adjoint to x, v, denoted by p, q. From the condition for the Hamiltonian to attain a maximum we obtain the expression for u

$$u^* = q|q|^{-1} \equiv \eta, \quad q \neq 0 \tag{1.2}$$

Let us consider the two-point boundary-value problem corresponding to the control u^* of (1.2). On partially solving the problem we obtain

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$$\dot{x}=v,\quad \dot{v}=\eta,\quad x(0)=x^0,\quad v(0)=v^0,\quad |x(t_f)|=1$$
$$p(t)\equiv p(t_f)=p^f,\quad q(t)=p^f(t_f-t),\quad p^f=\lambda x(t_f)|x(t_f)|^{-1} \tag{1.3}$$

The Lagrange multiplier λ, together with the constant vector p^f and minimum time t_f, are to be determined as a solution of the boundary-value problem (1.3).

It follows from (1.2) and (1.3) that whenever $x^0 \in S_1$ we have $|p^f| > 0$, that is, the control u^* is non-singular. Since the Hamiltonian is homogeneous, we can normalize the vector p^f by dividing it by $|p^f|$ so that $|p^f| = 1$. It then follows from (1.3) that $\lambda = \pm 1$, with $\lambda = 1$ if $x^0 \in B$, i.e. $|x^0| < 1$, and $\lambda = -1$ if $x^0 \overline{\in} B$, i.e. $|x^0| > 1$, where B is the interior of the unit sphere of the same dimensionality as the space, bounded by the sphere S_1. In addition, we infer from (1.3) that the control u^* of (1.2) is orthogonal to the plane touching the sphere at the point of intersection defined by the unit vector x^f, since $u^* = \lambda x^f, x^f = x(t_f), |x^f| = 1$.

Thus, we have constructed an optimal open-loop control, which turns out to be constant and of unit magnitude, while the trajectories are parabolas. The direction of the control vector is collinear ($x^0 \in B$) or anti-collinear ($x^0 \overline{\in} B$) with the unit vector x^f of the point at which the trajectory cuts the sphere, at time t_f. The quantities x^f and t_f are still undetermined (in all, there are n unknown constants).

2. SOLUTION OF THE BOUNDARY-VALUE PROBLEM OF THE MAXIMUM PRINCIPLE

To find the unknown parameters x^f and t_f, we integrate the Eqs (1.3) for x and v on the assumption that $u^* = \eta = p^f$. Equating $x(t_f) = x^f$, we solve this as an equation for η and note the equalities $\eta = p^f$, $|x^f| = 1$ and the transversality condition (1.3); this gives the expressions

$$\eta=-(x^0+v^0t_f)(t_f^2/2-\lambda)^{-1},\quad x^f=\eta/\lambda,\quad \lambda=\pm 1,\quad |x^0| \lessgtr 1 \tag{2.1}$$

where t_f is the unknown optimal time of motion, determined as the minimum positive root of the following fourth-degree algebraic equation

$$(t_f^2/2-\lambda)^2=l^2+2clh\,t_f+h^2t_f^2 \tag{2.2}$$
$$l=|x^0|,\quad h=|v^0|,\quad c=\cos(x^0,v^0)=(x^0,v^0)|x^0|^{-1}|v^0|^{-1}$$

Note that Eq. (2.2) follows immediately from (2.1), as can be verified by squaring one of the vector expressions. In addition, in the case of a sphere of "zero radius" (the point hits the origin) one obtains an equation of type (2.2) for t_f, in which one must formally put $\lambda = 0$ [1, 5].

Our next efforts will be directed at determining and investigating the minimum positive root t_f^* of Eq. (2.2). It depends on the values of three parameters

$$t_f^*=t_f^*(h,c,l),\quad h\geqslant 0,\quad -1\leqslant c\leqslant 1,\quad l\geqslant 0;\quad \lambda=\pm 1,\quad l \lessgtr 1 \tag{2.3}$$

A simple analysis shows that Eq. (2.2) always has a positive real root (and a negative one), i.e. the time-optimal problem is solvable. Note that there are possible situations in which two additional positive roots exist or one double root exists (see below, and also [5]).

It follows from the above constructions that this problem of motion of a point in the n-space is essentially equivalent to the two-dimensional problem ($n = 2$). The optimal trajectories lie in the plane spanned by the non-collinear vectors x^0, v^0 ($|c| < 1$). This conclusion follows obviously from the central symmetry of the control problem. It is corroborated by the fact that the solution is determined in all by the three quantities h, l and c. The case of motion along a straight line ($c = \pm 1$) turns out to be degenerate (critical).

Suppose that the minimum time t_f^* (2.3) has been determined analytically (see Section 3, e.g. by using Cardano's formulae) or numerically (Section 4). Then the required optimal control, in open-loop form $u_p^*(x^0, v^0)$, the unit vector of the intersection of the sphere x^{f*} and the phase trajectory $x^*(t)$, $v^*(t)$ are given by the following expressions

$$u_p^*=\eta^*=-(x^0+v^0t_f^*)(t_f^{*2}/2-\lambda)^{-1},\quad x^{f*}=\lambda\eta^*,\quad \lambda=\pm 1,\quad |x^0| \lessgtr 1$$
$$x^*=x^0+v^0t+\tfrac{1}{2}\eta^*t^2,\quad v^*=v^0+\eta^*t,\quad 0\leqslant t\leqslant t_f^* \tag{2.4}$$

Let us assume that the quantity t_f^* has been determined as a function of the vectors x^0 and u^0 in sufficiently large domains of possible values: $x^0 \in D_x \subseteq E^n, B \subset D_x, v^0 \in D_x, v^0 \in D_v \subseteq E^n$. Then formulae (2.3) and (2.4) define a solution of the synthesis problem in the domain $D_x \times D_v$, that is, the Bellman function of the problem θ and a feedback control u_s^*

$$\theta(x,v) = t_f^*(h,c,l), \quad h = |v|, \quad l = |x|, \quad c = \cos(x,v)$$
$$u_s^*(x,v) = -(x + v\theta)(\theta^2/2 - \lambda)^{-1}, \quad \lambda = \pm 1, \quad l \lesssim 1 \tag{2.5}$$

We will investigate the dependence of the roots t_f^* on the variables h, c and l in the neighbourhood of certain characteristic values of these motion parameters, which have an obvious geometrical and mechanical meaning.

3. LOCAL INVESTIGATION OF THE ANALYTIC PROPERTIES OF THE BELLMAN FUNCTION

We will consider approximate expressions and asymptotic expressions for θ as a function of arbitrary admissible values of the variables h, c, l and the discrete parameter λ. According to Section 2, $\theta(h, c, l, \lambda) = \min \arg Q(\theta) > 0$.

$$Q(\theta) \equiv (\theta^2/2 - \lambda)^2 - l^2 - 2clh\theta - h^2\theta^2 = 0 \tag{3.1}$$

3.1. *Behaviour of the Bellman function near the centre of the sphere.* Let us investigate the behaviour of the function $\theta(h, c, l, \lambda)$ for $0 \leqslant l \ll 1$ ($\lambda = 1$), using standard methods of perturbation theory [5–7]. Setting $l = 0$, we obtain from (3.1)

$$\theta_0(h) = \sqrt{2}(h^2 + 1 + ((h^2+1)^2 - 1)^{1/2})^{-1/2} \tag{3.2}$$

which does not depend on c. By (2.5), the optimal control u_s^* must point along the vector $v(\theta_0^2/2 < 1$ for $h > 0)$. The case $h = 0$ ($l = 0$) is singular: the control may point in any direction if $l = h = 0$, as is obvious. As time elapses and $h > 0, l > 0$, the ambiguity vanishes; the control, velocity and position vectors will be collinear (no allowance is being made for perturbations). Note that the function $\theta_0(h)$ (3.2) (and also $\theta(h, c, l, l)$) tend to zero like h^{-1} as $h \to \infty$ (see Sections 3 and 4).

Let $h > 0$ ($h \sim 1$); then the root θ_0 (3.2) is simple and one has the following representation for θ

$$\theta(h,c,l,1) = \theta_0(h) + l\theta_1 + l^2\theta_2 + l^3 \ldots, \quad \theta_1 = 2chd^{-1}$$
$$\theta_2 = (2ch + \theta_1(h^2+1))\theta_1(\theta_0 d)^{-1}, \ldots, \quad d = -2h(h^2+2)^{1/2} \tag{3.3}$$

If $h > 0$, then $d < 0$, and expansions (3.3) are absolutely and uniformly convergent for sufficiently small $l > 0$. The required root θ may also be determined by successive approximations. The coefficients are such that $\theta_i \sim h^{-1}$ as $h \to \infty$, which is obvious.

Let us consider an almost degenerate situation. The following subcases are of interest: (a) $h \sim l$, (b) $h \sim l^{1/2}$ ($l \ll h \ll 1$), and (c) $h \sim l^2$ ($0 < h \ll l$). Consider the first, i.e. $l = \varepsilon L$, $h = \varepsilon H$, where $0 < \varepsilon \ll 1$, $L, H \sim 1$; then we find that

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3 \ldots, \quad \theta_0 = \sqrt{2}, \quad \theta_1 = -(L^2 + 2cLH\theta_0 + H^2\theta_0^2)^{1/2}\theta_0^{-1}$$
$$\theta_2 = (cLH + H^2\theta_0)\theta_0^{-2} - \tfrac{1}{2}\theta_1^2\theta_0^{-1}, \ldots \quad (\theta_1 \neq 0) \tag{3.4}$$

The quantities θ_0 and θ_1 obtained above have simple mechanical meanings and need no comment. Not so the case $\theta_1 = 0$, which needs further analysis; but that situation arises only when $c = -1$, $\sqrt{(2)}H = L$. The construction of the solution is also elementary; if $\theta_1 \sim \varepsilon$, then $\theta = \theta_0 + \varepsilon^2\theta_2 + \varepsilon^3 \ldots$ for $\theta_2 \neq 0$ (the case $\theta_1 = \theta_2 = 0$ occurs when $L = 0, H = 0$).

It follows from an analysis of formulae (3.3) for the coefficients θ_i that these expansions are also valid in subcase (b), i.e. $h = \varepsilon^{1/2}H$, $l = \varepsilon L$. Strictly speaking, however, the small parameter here is $\varepsilon^{1/2}$.

Let us expand θ in powers of ε in subcase (c), when $h = \varepsilon^2 H$, $l = \varepsilon L$. Applying a standard procedure of perturbation theory, we obtain the required expression of type (3.4)

$$\theta_0 = \sqrt{2}, \quad \theta_1 = -L/\sqrt{2}, \quad \theta_2 = -\tfrac{1}{4}L^2/\sqrt{2} - 2cH, \ldots$$

The physical meaning of the coefficients θ_i ($i = 0, 1, 2, \ldots$) is fairly clear and needs no comment. This completes the investigation of the Bellman function θ when the point is situated near the centre of the sphere (the origin); see Section 4.

3.2. *Asymptotic investigation of the Bellman function at great distances*. Let us consider the reverse situation, when $l \gg 1$ ($\lambda = -1$). It follows from an analysis of Eq. (3.1) that $\theta \sim l^{1/2}$; a good small parameter will be $\mu = l^{-1/2}$. The expansions for θ are asymptotic expansions: $\theta = \mu^{-1}\theta_{-1} + \theta_0 + \mu\theta_1 + \mu^2 \ldots$. Standard methods yield the unknown coefficients θ_i; as a result we obtain the required expression

$$\theta = \sqrt{2}l^{\frac{1}{2}} + ch + (4h^2 - c^2h^2 - 2)(8l)^{-\frac{1}{2}} + O(h^3/l) \tag{3.5}$$

The first term in (3.5) is the (approximate) duration of the motion, the second is a compensation for the distance from the centre of the sphere, the third is a correction obtained by allowing for the dimension of the sphere. Formula (3.5) is adequate for estimates at $h \sim 1$; if necessary, one can determine further terms of the expansion. The situation in which $h \gg 1$ (and $l \gg 1$) requires a special study, which will be carried out below, in Section 3.5.

3.3. *Behaviour of the Bellman function near a state of rest*. Let us consider the case of low velocity $0 \leqslant h \ll 1$, on the assumptions that $l \sim 1$. The situation $l \ll 1$ and $l \gg 1$ has already been considered (Sections 3.1 and 3.2). We will seek θ as expansions in powers of the small parameter h, as in the case of (3.3) and (3.4)

$$\theta = \theta_0 + h\theta_1 + h^2\theta_2 + h^3 \ldots, \quad \theta_0 = \sqrt{2}|l-1|^{\frac{1}{2}}, \quad l \neq 1, \quad \theta_1 = \mp c \tag{3.6}$$

$$\theta_2 = (\mp 2c^2 l + (2 - 3c^2)|l-1| \pm c^2)d^{-1}, \quad d = \mp 2\sqrt{2}l|l-1|^{\frac{1}{2}} \neq 0$$

The upper signs in these expressions for θ_1 and θ_2 correspond to the condition $0 < l < 1$, and the lower signs correspond to $l > 1$, where $l \sim 1$ and $|l-1| \sim 1$; the situation $|l-1| \sim \varepsilon \ll 1$ will be studied below, in Section 3.6. The kinematic meaning of the coefficients θ_i (3.6) is sufficiently clear; θ_0 corresponds to the duration of motion to the sphere along a straight line from zero initial velocity, and θ_1 makes allowance for the magnitude and direction of the velocity, which is asymptotically small; the coefficient θ_2 is more complicated to treat, as it incorporates some rather subtle effects.

3.4. *Asymptotic analysis of the Bellman function at high velocities*. In the case $h \gg 1$ the function θ may have both regular ($\theta \sim h^{-1}$) and singular ($\theta \sim h$) representations. The first possibility clearly occurs when $l < 1$ or $l > 1$ under the conditions $-1 \leqslant c < 0$, $|s|l \leqslant 1$; these conditions have an obvious geometrical meaning. But if $l > 1$ and $|s|l > 1$ ("missed target"), one has the second possibility (singular expansions). The case $c < 0$, $0 < 1 - s^2l^2 \ll 1$, where $s^2 = 1 - c^2$, requires a separate study.

For convenience we will introduce a small parameter $\xi = h^{-1}$ and transform Eq. (3.1) to the form

$$\xi^2(\theta^2/2 - \lambda)^2 = \xi^2 l^2 + 2\xi c l\theta + \theta^2, \quad l \neq 1 \tag{3.7}$$

Considering the first possibility, we construct regular expansions in powers of ξ

$$\begin{aligned} &\theta = \theta_0 + \xi\theta_1 + \xi^2\theta_2 + \xi^3\theta_3 + \xi^4 \ldots, \quad \theta_0 = 0, \quad \theta_1 = \theta_1^\pm = -cl \pm d \\ &\theta_2 = \theta_2^\pm = \mp \tfrac{1}{2}(\theta_1^\pm)^2 d^{-1}, \quad \theta_3 = \theta_3^\pm = \tfrac{1}{2}(\tfrac{1}{4}(\theta_1^\pm)^4 \mp \\ &\mp \theta_1^\pm\theta_2^\pm - (\theta_2^\pm)^2)d^{-1}, \quad d = (1 - s^2l^2)^{\frac{1}{2}} \end{aligned} \tag{3.8}$$

The upper signs correspond to $\lambda = 1$, i.e. $l < 1$, and $\lambda = -1$, i.e. $l > 1$, respectively. The behaviour of the coefficients θ_i depends radically on the quantity s^2l^2, as already pointed out. If $d^2 > 0$, which is the case when $x \in B$ ($l < 1$), the regular expansions (3.8) hold and yield the required representation of the Bellman function. When $l > 1$ the expansion is again regular if $c < 0$ and $d^2 > 0$. In the critical case $d^2 = 0$ the expansions are still valid. The expressions for θ_i are then

$$\theta_0 = 0, \quad \theta_1 = -cl, \quad \theta_2 = (-cl)^{\frac{1}{2}}, \quad \theta_3 = \tfrac{1}{2}, \ldots \tag{3.9}$$

In the near-critical case $d^2 = O(\xi) > 0$ one can apply the expansions (3.8). But if $d^2 = O(\xi) < 0$, the regular expansions are no longer valid; this is again a case of a "missed target". When $d^2 = 1 - s^2l^2 = -a^2\xi^2$, regular expansions are possible if in addition $a^2 < 1$, in particular, when $a^2 = O(\xi)$.

Let us consider the case of a singular representation of θ when $|sl| > 1$ or $c \geqslant 0$. We obtain

$$\begin{gathered}\theta = \xi^{-1}\theta_{-1} + \xi\theta_1 + \xi^3\theta_3 + \xi^5 \ldots, \quad \theta_{-1} = 2 \\ \theta_1 = 2cl - 1, \quad \theta_3 = l^2 - 1 + (1 - 8cl)(2cl - 1), \ldots\end{gathered} \tag{3.10}$$

The first term in (3.10) corresponds to the total time of deceleration and re-entry to the sphere; the velocity of approach is of the order of $h = \xi^{-1}$. The small increments $O(\xi)$, etc. characterize delicate effects of nearly time-optimal manoeuvres.

3.5. *Asymptotic behaviour of the Bellman function at great distances and high velocities.* It is now natural to examine a situation in which $l, h \gg 1$. To fix our ideas, let us first take quantities of the same order of magnitude: $l = \varepsilon^{-1}L$, $h = \varepsilon^{-1}H$, where $0 < \varepsilon \ll 1$ is a small parameter. Equation (3.1), which defines the function θ, becomes

$$L^2 + 2cLH\theta + H^2\theta^2 = \varepsilon^2(\theta^2/2 + 1)^2, \quad L, H \sim 1 \tag{3.11}$$

The unknown function θ admits of both regular and singular representations as expansions in powers of ε, which will now be constructed. Regular expansions may occur if $s = \varepsilon S$, $S \sim 1$, where s is the size of the angle between the vectors x and v. Standard methods yield

$$\begin{gathered}\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3 \ldots, \quad \theta_0 = LH^{-1} \\ \theta_1 = -((\theta_0^2/2 + 1)^2 - S^2\theta_0^2)^{1/2}, \quad \theta_2 = \theta_0 H^{-2}(\theta_0^2/2 + 1) - \tfrac{1}{2}\theta_0 S^2), \ldots\end{gathered} \tag{3.12}$$

It is assumed here that the expression for θ_1 is defined (real) and strictly negative; in the case $\theta_1 = 0$ one must also analyse this expression. If s is small to a higher order, say $s = \varepsilon^2 S$, a solution in the form of (3.12) always exists.

We will now consider singular expansions of θ, which occur when $s \sim 1$. Put $\theta = \varepsilon^{-1}\Theta$; we have the following equation for the unknown $\Theta \sim 1$

$$\tfrac{1}{4}\Theta^4 + \varepsilon^2\Theta^2 + \varepsilon^4 = \varepsilon^2L^2 + 2\varepsilon cLH\Theta + H^2\Theta^2 \tag{3.13}$$

Equation (3.13) is practically identical with the analogue of (3.7) for the "missed target" case ($s^2l^2 > 1$); the required expansion $\Theta = 2H + \varepsilon cLH^{-1} + \varepsilon^2 \ldots$ is found in the same way.

Now suppose that l and h are of different orders of magnitude. The interesting situation is $l = \varepsilon^{-2}L$, $h = \varepsilon^{-1}H$. Obviously, θ will be asymptotically large, i.e. $\theta = \varepsilon^{-1}\Theta$. Substituting this expression into Eq. (3.1), we reduce it to the form

$$\tfrac{1}{4}\Theta^4 + \varepsilon^2(\Theta^2 + \varepsilon^2) = L^2 + 2cLH\Theta + H^2\Theta^2 \tag{3.14}$$

Apart from terms $O(\varepsilon^2)$, this equation is identical with that investigated in [5]. It transforms formally to an expression containing no perturbations

$$\begin{gathered}\tfrac{1}{4}\Theta^4 = L^{*2} + 2c^*L^*H^*\Theta + H^{*2}\Theta^2 \\ L^{*2} = L^2 - \varepsilon^4, \quad H^{*2} = H^2 - \varepsilon^2, \quad c^* = c(L/L^*)(H/H^*)\end{gathered} \tag{3.15}$$

One must bear in mind here that $L^2, H^2 \sim 1$, i.e. L^* and H^* are defined, and the quantity c^* varies between limits that are greater, by quantities $O(\varepsilon^2)$, than c $(-1 \leqslant c \leqslant 1)$. A complete numerical solution has been obtained for Eq. (3.15) [5]. One can thus determine the solution of Eq. (3.14). Note that the same equation is obtained in the case of a sphere of asymptotically small radius with a different normalization.

The reverse situation, when $l = \varepsilon^{-1}L$, $h = \varepsilon^{-2}H$, is basically similar to that investigated in Section 3.4.

3.6. *Behaviour of the Bellman function near the sphere surface.* Let us consider an approximate expression for θ at $l = 1 + \varepsilon$, where $\varepsilon \lessgtr 0$, $|\varepsilon| \ll 1$, i.e. near the sphere. Substituting the value of l into Eq. (3.1), we obtain

$$\theta^4 - 4\lambda\theta^2 = 8ch\theta + 4h^2\theta^2 + 4\varepsilon(2+\varepsilon+2ch\theta)$$
$$\lambda = \pm 1, \quad \varepsilon \lessgtr 0; \quad \theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3 \ldots \tag{3.16}$$

The solution of the equation according to (3.16) will be sought as a series in integer powers of ε, on the assumption that $c \sim 1, h \sim 1$. When $\varepsilon = 0$ we obtain the following expressions for the unknown θ_0

$$\theta_0 P(\theta_0) = 0, \quad P(\theta) \equiv \theta^3 - 4(\lambda + h^2)\theta - 8ch \tag{3.17}$$
$$\theta_0 = 0, \quad \theta_0 = \min \arg P(\theta) > 0$$

These expressions admit of two possibilities, both of which will be considered. The first root (zero) leads to expansions

$$\theta_0 = 0, \quad \theta_1 = -(ch)^{-1}, \quad \theta_2 = -\tfrac{1}{2}(\lambda + h^2)(ch)^{-3} + \tfrac{1}{2}(ch)^{-1}, \ldots \tag{3.18}$$

The principal term in expansion (3.16) and (3.18) will be $\varepsilon\theta_1 = -\varepsilon(ch)^{-1}$. Since $\theta > 0$, the condition that $\varepsilon\theta_1$ should be positive yields the inequality $\varepsilon C < 0$. Thus, for $\varepsilon < 0$ ($l < 1$, $\lambda = 1$), the vector v must be directed out of the sphere ($c > 0$); conversely, when $\varepsilon > 0$ ($l > 1$, $\lambda = -1$), v must be directed into the sphere ($c < 0$).

Let us consider the critical case, when $ch = \varepsilon\gamma$, $\gamma \sim 1$. In the general case the expansions are in powers of $|\varepsilon|^{1/2}$

$$\theta = |\varepsilon|^{1/2}(2(1 \mp \gamma)/(1-h^2))^{1/2} + |\varepsilon| \ldots, \quad h^2 \neq 1, \quad \varepsilon \lessgtr 0, \quad \lambda = \pm 1 \tag{3.19}$$

and they exist if the expression for θ_1 is real. The other critical cases may be studied similarly.

Let us investigate the other situation, when the velocity at points inside the sphere ($l < 1$) is directed inwards ($c < 0$), while that of points outside the sphere ($l > 1$) is directed outwards ($c > 0$). The unknown θ_0 in (3.16) is determined by the second expression in (3.17).

We will now investigate the roots of the cubic equations (3.17). We have to find the minimum positive root θ_0 as a function of the parameters h, c and $\lambda = \pm 1$. We first use Cardano's formulae, solving for $h = h(\theta, c, \lambda)$

$$h = -c\theta^{-1} \pm (c^2\theta^{-2} + \theta^2/4 - \lambda)^{1/2}, \quad c \lessgtr 0, \quad \lambda = \pm 1 \tag{3.20}$$

It follows from (3.20) that if $c < 0$, $\lambda = 1$ ($l < 1$), the family of curves $\theta(\eta, c, 1)$ consists of two non-monotone families of branches (the $\pm$ signs in (3.20)), while when $c > 0$, $\lambda = -1$ ($l > 1$) the curve $\theta(h, c, -1)$ consists of one monotone increasing family of branches (with plus sign; the minus sign leads to $h < 0$). It is assumed here that h is the argument and c is the parameter of the families.

We will now apply Cardano's formulae directly to the equation $P(\theta_0) = 0$ of (3.17). As usual, we define

$$p = -4(\lambda + h^2), \quad q = -8ch, \quad d = q^2/4 + p^3/27 \tag{3.21}$$

There are three possibilities. Let $d < 0$, which is surely the case when $-1 < c < 0$, $\lambda = 1$ ($l < 1$). Then the equation admits of two positive roots (and one negative one), which are calculated by the formula

$$\theta_0^{(k)} = (-q/2 \pm j|d|^{1/2})^{1/3} + (-q/2 \mp j|d|^{1/2})^{1/3} \gtrless 0, \quad j = \sqrt{-1}$$
$$k = 1,2,3, \quad \theta_0^{(k)} = 2\mu_k, \quad \mu_k = \operatorname{Re}(-q/2 + j|d|^{1/2})^{1/3} \tag{3.22}$$

Of the two positive roots θ_0^k, choose the least. Note that $d < 0$ is also true when $0 < c < 1$ ($\lambda = -1$, $l > 1$), if $h \geqslant 2$, and when $1 < h < 2$ if $0 < c < f(h)$, where $f(h) \equiv (4/27)^{1/2}(h^2-1)^{3/2}h^{-1}$. In this situation Eq. (3.17) admits of one positive and two negative roots, which are calculated by formulae (3.22).

Let $d = 0$, which is the case only when $c = -1$ for $h = 1/\sqrt{2}$ ($\lambda = 1, l < 1$, the point is inside the sphere) and when $0 < c < 1, c = f(h)$ for $1 < h < 2$ ($\lambda = -1, l > 1$, the point is outside the sphere). In that case Eq. (3.17) has three real roots, two of which coincide

$$\theta_0^{(1)} = -2(q/2)^{1/3}, \quad \theta_0^{(2,3)} = (q/2)^{1/3}, \quad q = -8ch \tag{3.23}$$

As follows from (3.21), in the first situation ($c = -1$) the double root $\theta_0^{(2,3)}$ (3.23) is positive, while in the second ($c = f(h)$) the required root is $\theta_0^{(1)}$ (3.23).

Now let $d > 0$, which may occur only when $1 < h < 2$ for $1 > c > f(h) > 0$ ($\lambda = -1, l > 1$, a point outside sphere) and for $0 < h < 1, 0 < c < 1$. Then Eq. (3.17) admits of one positive real root θ_0

$$\theta_0 = (-q/2 + d^{1/2})^{1/3} + (-q/2 - d^{1/2})^{1/3}, \quad q = -8ch < 0 \tag{3.24}$$

These expressions for the roots may also be formulated in trigonometric notation. They form the basis, via perturbation methods, for the construction of regular expansions in powers of ε in all cases other than the critical, which corresponds to a double root $\theta_0^{(2,3)}$ (3.23) when $c = -1$. In the latter case, Eq. (3.1) (as well as (3.16)) can be solved exactly by elementary analytical means (see Section 3.9).

The required expansions are

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3 \ldots, \quad \theta_0 = \theta_0(h, c, \lambda), \theta_1 = 8(1 + ch\theta_0)(P_0 + \theta_0 P_0')^{-1} \tag{3.25}$$

$$\theta_2 = (8ch\theta_1 + 4 - (P_0' + {\textstyle\frac{1}{2}}\theta_0 P_0'')\theta_1^2)(P_0 + \theta_0 P_0')^{-1}, \ldots, \quad P_0 \equiv P(\theta_0)$$

where $\theta_0 = 0$ or $\theta_0 = \min_k \theta_0^{(k)} > 0$ according to (3.17) and the computations (3.22)–(3.24). Note that special expressions for $\theta_0 = 0$ already appear in (3.18) and (3.19). The main meaningful comment concerns the principal terms of expansions (3.18), (3.19) and (3.25); this comment was made previously.

3.7. *Behaviour of the Bellman function when the radius vector and velocity vector are orthogonal to each other.* We will now investigate the situation in which $|c| \ll 1$ but $l \sim 1, h \sim 1$. Equation (3.1) reduces at $c = 0$ to a biquadratic equation, whose solution is

$$\theta_0 = \sqrt{2}(h^2 + \lambda \mp ((h^2 + \lambda)^2 + l^2 - 1)^{1/2})^{1/2}, \quad \lambda = \pm 1, \quad l \lessgtr 1 \tag{3.26}$$

Note that the function θ_0 (3.26) is not continuous as $l \to 1 \pm 0$; we will assume from now on that $l \lessgtr 1$ (the case $l = 1 \pm \varepsilon$ has already been studied). Under this condition θ_0 is a simple root, and we obtain the following regular expansion for the quantity $z = \theta^2$

$$\begin{gathered} z = z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 \ldots, \quad z_0 = \theta_0^2, \quad z_1 = 2lh\theta_0 d^{-1} \\ z_2 = 2l^2h^2d^{-2} - l^2h^2\theta_0^2 d^{-3}, \quad d = \mp((h^2+1)^2 + l^2 - 1)^{1/2}, \quad l \lessgtr 1 \end{gathered} \tag{3.27}$$

Expression (3.26) for θ_0 is fairly cumbersome and difficult to interpret. Nevertheless, its limiting values at $l \to 0, 1, \infty$ and/or as $h \to 0, \infty$ correspond to intuitive expectations. The next increment cz_1 allows for an increase or decrease in the response time, depending on the sign of the parameter c and the position of the point ($l \lessgtr 1$).

3.8. *Investigation of the Bellman function near the degenerate case of almost collinear position and velocity vectors.* We will now construct an approximate solution of Eq. (3.1) in the case $c = 1 - \varepsilon \ll 1$. The root values when $\varepsilon = 0$ ($c = -1$) are

$$\theta_0 = \mp h + (h^2 \mp 2(l-1))^{1/2}, \quad l \lessgtr 1 \tag{3.28}$$

This expression corresponds to one-dimensional motion and needs no explanation. In the general position, the root θ_0 (3.28) is simple, and correction of this value for $\varepsilon > 0$ is achieved by standard means

$$\begin{gathered} \theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3 \ldots, \quad \theta_0 = \theta_0(h, l), \quad \theta_1 = -2lh\theta_0 d^{-1} \\ \theta_2 = ((h^2 + \lambda - 3\theta_0^2/2)\theta_1^2 - 2lh\theta_1)d^{-1}, \ldots, \quad d = \theta_0^3 - 2(h^2 + \lambda)\theta_0 - 2lh \end{gathered} \tag{3.29}$$

The fact that there is a small angle α ($\alpha^2/2 = \varepsilon$) between the vectors causes an increase ($l < 1$) or decrease ($l > 1$) in the optimal time of motion, as is intuitively obvious.

3.9. *Analysis of the Bellman function near the degenerate case of almost anti-collinear position and velocity vectors.* Proceeding by analogy with the previous case, let us consider the situation $c = -1 + \varepsilon$, $0 \leqslant \varepsilon \ll 1$. Setting $\varepsilon = 0$, we obtain

$$\theta_0 = -\beta h + \gamma(h^2 + 2(\lambda + \beta l))^{1/2} \tag{3.30}$$

Here the signs in $\gamma = \pm 1$, $\beta = \pm 1$ are independent. Analysis shows that for $l < 1$ ($\lambda = 1$) the quantity θ_0 is described by two expressions

$$\begin{aligned} &\theta_0 = h + (h^2 + 2(1 - l))^{1/2}, \quad 0 < h^2 \leqslant l^2/2 \\ &\theta_0 = -h + (h^2 + 2(1 + l))^{1/2}, \quad h^2 > l^2/2 \end{aligned} \tag{3.31}$$

If the point is outside the sphere ($l > 1$, $\lambda = -1$), the response time is always described by one expression

$$\theta_0 = -h + (h^2 + 2(l - 1))^{1/2}, \quad h > 0 \tag{3.32}$$

These expressions are elementary and need no explanation. We need only note that the expressions (3.31) do not join up smoothly, but the value of $\theta_0 \equiv \sqrt{2}$ along the "seam" is constant, that is, is independent of l and h ($l < 1$); see Section 4.

The function θ in the situation of the general position is corrected by standard means

$$\begin{aligned} &\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3 \ldots, \quad \theta_0 = \theta_0(h, l), \quad \theta_1 = 2lh\theta_0 d^{-1} \\ &\theta_2 = ((h^2 + \lambda - 3\theta_0^2/2)\theta_1^2 + 2lh\theta_1)d^{-1}, \ldots, \quad d = \theta_0^3 - 2(h^2 + \lambda)\theta_0 + 2lh \end{aligned} \tag{3.33}$$

The quantities θ_0 in (3.33) are defined as in (3.31) and (3.32). The expressions for θ_i and d in these expansions differ from the corresponding expressions in (3.29) only in the sign of $2lh$.

In this section we have presented a fairly detailed investigation of the analytical properties of the Bellman function in the neighbourhood of states where an exact analytical solution is possible. We have also presented an asymptotic analysis for large values of the parameters of motion ($l, h \to \infty$). It is of considerable interest to compare these analytic results with the results of a fairly comprehensive numerical solution of the optimal control problem.

4. RESULTS OF NUMERICAL SIMULATION

A direct numerical solution of the fourth-degree algebraic equation (3.1) for $\theta = \theta(h, c, l, \lambda)$ and the choice of the minimum positive value for the unbounded domain of the parameters h and l is an extremely laborious task. The function θ may be represented as a family of surfaces in the first octant of the three-dimensional space (θ, h, l), for which $-1 \leqslant c \leqslant 1$ is the parameter of the family, $\lambda = \pm 1$. However, the construction of the family involves very cumbersome and non-intuitive results, because of the difficulty of projecting the surfaces. Preliminary analysis shows that it is preferable to define the family of curves (θ, h) with parameter c for a comparatively small number of values of the parameter l.

The construction of the functions $\theta(h)$ for fixed c and l is simplified, reducing to explicit algebraic expressions, if the quadratic equation (3.1) is solved for $h = h(\theta, c, l, \lambda)$. We have [5]

$$\begin{aligned} &h = \tfrac{1}{2}\theta^{-2}(-2\theta c l + \gamma(4\theta^2 c^2 l^2 + 4\theta^2((\theta^2/2 - \lambda)^2 - l^2))^{1/2}) \\ &h \geqslant 0, \quad \theta > 0, \quad l \lessgtr 1, \quad \lambda = \pm 1, \quad -1 \leqslant c \leqslant 1, \quad \gamma = \pm 1 \end{aligned} \tag{4.1}$$

Instead of the unknown functions $\theta(h)$, the inverse function $h(\theta)$ given by this expression can be investigated analytically and numerically. Mathematical simulation using formula (4.1) produced a graphical solution $\theta(h, c, l)$ corresponding to the analysis of the behaviour of the Bellman function carried out in Section 3. We will now present and investigate a few typical families of curves (varying c as parameter) for different values of $l \geqslant 0$.

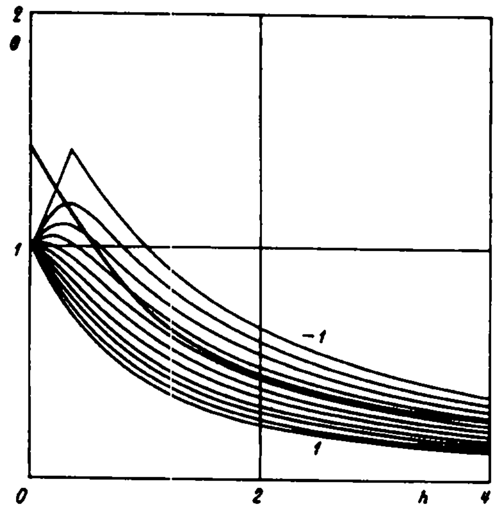

Fig. 1.

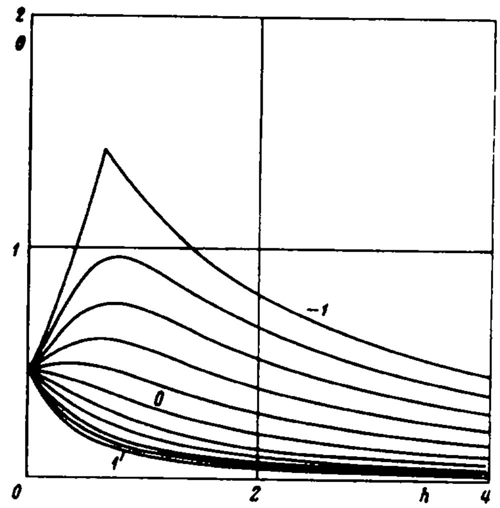

Fig. 2.

We first consider the situation $0 \leqslant l \ll 1$, represented in Figs 1 and 2. Figure 2 shows the family of curves for $c_i = -1 + 0.2i$, $i = 0, 1, \ldots, 10$ (with stepsize $\Delta c = 0.2$) and two values of l: $l = 0$ (Section 3.1) and $l = 0.5$ (the "general" case). When $l = 0$ the family is independent of c (see (3.2)) and merges into a single curve issuing from the point $\theta_0 = \sqrt{2}$ at $h = 0$. The behaviour of the curves for $0 \leqslant l \ll 1$ is in perfect agreement with the analytic results of Section 3. Let us examine the families of curves for different values of l. The general point θ from which the curves of each family issue for $h \geqslant 0$ ascends to the value $\theta = \sqrt{2}$, while the family begins to narrow down ("merging") as $l \to 0$, tending to the aforementioned single curve (3.2). As $l \to 1$ the family begins to spread out and to fill the strip $\theta \leqslant \theta \leqslant \sqrt{2}$, and the point from which the family issues is such that $\theta \to 0$ as $l \to 1$ (see Fig. 2). The analysis of Section 3.6 is in complete agreement with the situation $l = 1 - \varepsilon$; see also Section 3.8 ($c = -1 + \varepsilon$) and Section 3.9 ($c = 1 - \varepsilon$), $0 < \varepsilon \ll 1$. Thus, inside the sphere $x \in B$, the families of curves and the Bellman function vary fairly monotonically and smoothly, except in the degenerate case $c = -1$, which produces corner points. The change in the Bellman function is also quite intensive near $h = 0$ and as $l \to 1$ (the irregularity of θ becomes worse up to the discontinuity).

We will now consider the results of the simulation for $x \bar{\in} B$ ($l > 1$), see Figs 3–5. Let us begin with a case in which the point is near the sphere, say, $l = 1.1$ (Fig. 3); this family of curves corresponds to the analysis in Section 3.6. The curves do not represent the single-valued functions, because, for technical reasons, the figures include all roots, not only the minimum ones (the four curves on the upper right

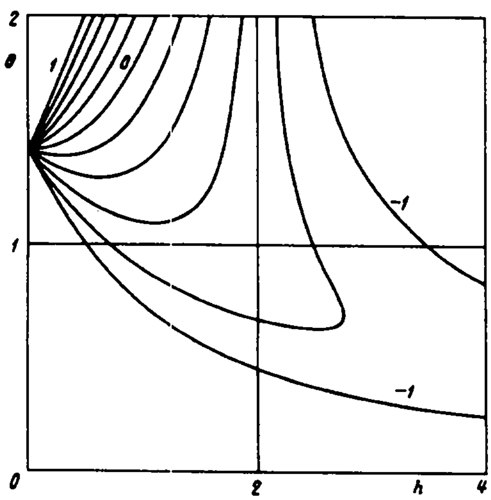

Fig. 3.

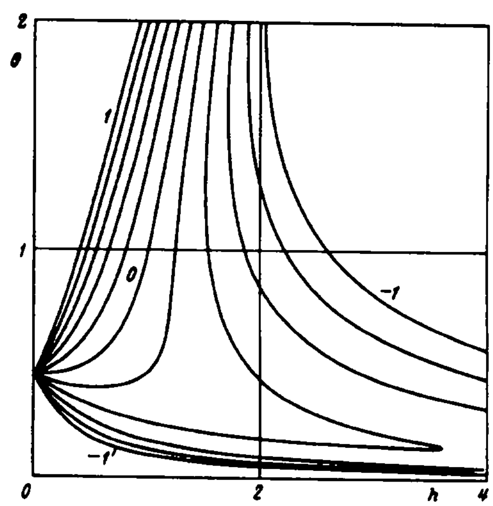

Fig. 4.

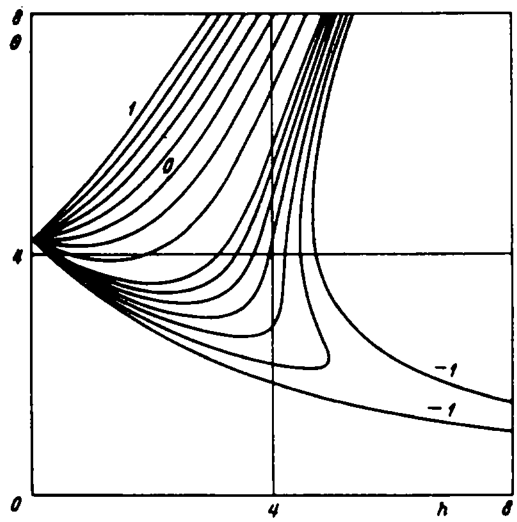

Fig. 5.

should be removed). Note that in the neighbourhood of $c = -0.4$ there is a marked change in the response time as a function of c and h, because of the possibility of a "miss". Attention should be drawn to the behaviour of the family of curves vertically upwards: they depart upwards, pointing at and approaching the straight line $\theta \sim 2h$, $h \to \infty$; see Section 3.4 and formula (3.10).

Figure 4 depicts the "general" situation $l > 1$, $l \sim 1$ (in the figure, $l = 2$). The family of curves shown confirms the analysis for $h \ll 1$, $h \gg 1$, $c = 1 + \varepsilon$, $c = \varepsilon$, where $0 < |\varepsilon| \ll 1$. Attention should be drawn to the jump in the optimal time of motion when a "miss" occurs at $c = -0.8$. As already noted, the family departs upwards to the right, as in (3.10).

Let us now consider the case when $l \gg 2$, see Sections 3.2 and 3.5, in particular, $l = 10$ (Fig. 5). The family of curves is similar to that constructed for a sphere of asymptotically small radius (the point hits the origin [5]). At values of c close to $c = -1$ (see Section 3.9), a more densely packed family is given: $c = -1 + 0.05i$, $i = 0, 1, \ldots, 6$, then $\Delta c = 0.3$, and then $\Delta c = 0.2$ from $c = -0.4$ to $c = 1$. One should note the very subtle difference between the case of a sphere of non-zero radius and the limiting case of a point. Near values $c = -1$ there is a narrow family of curves that decrease monotonically as $h \to \infty$, i.e. without any sudden change in the response time. Outside this thin layer a range of values $c < 0$ exists that permits misses and non-monotone behaviour of θ as a function of h (for fixed c) and discontinuity of the Bellman function. Then comes a range of values (from a few negative values to $c = 1$) that produce θ as a monotone smooth function of h ("flight with return").

Thus, we have carried out a fairly detailed construction and investigation of the Bellman function for the problem of the time-optimal approach of a point mass to a sphere (intersection of the sphere) of finite radius, under the action of a force of limited magnitude. We have established that, depending on the values of the phase variables, the Bellman function has certain properties of non-smoothness and discontinuity; we have determined the nature of these irregularities. We have in fact constructed a picture of the synthesis (see Section 2). Note that this solves the problem of a point approaching a cylindrical subspace, in particular, a hyperplane.

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REFERENCES

1. PONTRYAGIN L. S., BOLTYANSKII V. G., GAMKRELIDZE R. V. and MISHCHENKO Ye. F., *Mathematical Theory of Optimal Processes*. Nauka, Moscow, 1969.
2. KRASOVSKII N., *Game-theoretical Problems of the Approach of Motions*. Nauka, Moscow, 1970.
3. KRASOVSKII N. N. and SUBBOTIN A. I., *Positional Differential Games*. Nauka, Moscow, 1974.
4. GRODZOVSKII G. L., IVANOV Yu. N. and TOKAREV V. V., *Mechanics of Space Flight. Optimization Problems*. Nauka, Moscow, 1975.

5. AKULENKO L. D., The perturbed time-optimal problem of the control of the final position of a point mass by means of a bounded force. *Prikl. Mat. Mekh.* **58**, 2, 12–21, 1994.
6. GOURSAT E., *Cours d'Analyse*. Vol. II, Pt. 1. Paris, 1970.
7. YERUGIN N. P., *Implicit Functions*. Izd. Leningrad. Gos. Univ., Leningrad, 1956.

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